

# HIGHER CENTRAL EXTENSIONS IN MAL'TSEV CATEGORIES

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**ABSTRACT.** Higher dimensional central extensions of groups were introduced by G. Janelidze as particular instances of the abstract notion of covering morphism from categorical Galois theory. More recently, the notion has been extended to and studied in arbitrary semi-abelian categories. In this article, we further extend the scope to exact Mal'tsev categories and beyond.

## 1. INTRODUCTION

A *Galois structure*  $\Gamma = (\mathbb{C}, \mathbb{X}, H, I, \eta, \mathcal{E})$  [19, 24] consists of a category  $\mathbb{C}$ , a full replete reflective subcategory  $\mathbb{X}$  of  $\mathbb{C}$  with inclusion functor  $H: \mathbb{X} \rightarrow \mathbb{C}$ <sup>1</sup>, left adjoint  $I: \mathbb{C} \rightarrow \mathbb{X}$  and reflection unit  $\eta_A: A \rightarrow HI(A)$  (for  $A \in \mathbb{C}$ ), and a class  $\mathcal{E}$  of morphisms in  $\mathbb{C}$  such that

- (E1)  $\mathcal{E}$  contains all isomorphisms;
- (E2)  $\mathcal{E}$  is pullback-stable, in the following sense: the pullback of a morphism in  $\mathcal{E}$  along any morphism in  $\mathbb{C}$  *exists* and lies in  $\mathcal{E}$ ;
- (E3)  $\mathcal{E}$  is closed under composition;
- (G)  $HI(\mathcal{E}) \subseteq \mathcal{E}$ .

Given a fixed Galois structure  $\Gamma = (\mathbb{C}, \mathbb{X}, H, I, \eta, \mathcal{E})$ , and for any object  $B \in \mathbb{C}$ , we shall use the notation  $(\mathbb{C} \downarrow B)$  for the full subcategory of the slice category  $\mathbb{C}/B$  containing those  $f \in \mathbb{C}/B$  which lie in the class  $\mathcal{E}$ . We shall often write  $(A, f)$  instead of just  $f$  in order to denote an object  $f: A \rightarrow B$  of  $(\mathbb{C} \downarrow B)$ . A *monadic extension* (or *effective  $\mathcal{E}$ -descent morphism*) is an object  $(E, p) \in (\mathbb{C} \downarrow B)$  such that the pullback functor  $p^*: (\mathbb{C} \downarrow B) \rightarrow (\mathbb{C} \downarrow E)$  is monadic. A *trivial covering* of  $B$  is an object  $(A, f) \in (\mathbb{C} \downarrow B)$  such that the induced commutative square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ f \downarrow & & \downarrow HI(f) \\ B & \xrightarrow{\eta_B} & HI(B) \end{array}$$

is a pullback. If  $(E, p) \in (\mathbb{C} \downarrow B)$  is a monadic extension and  $(A, f) \in (\mathbb{C} \downarrow B)$ , then  $(A, f)$  is said to be *split* by  $(E, p)$  if  $p^*(A, f)$  is a trivial covering of  $E$ . The full subcategory of  $(\mathbb{C} \downarrow B)$  containing those  $(A, f) \in (\mathbb{C} \downarrow B)$  that are split by  $(E, p)$  is denoted by  $\mathbf{Spl}_\Gamma(E, p)$ . A *covering* of  $B$  is an object  $(A, f) \in (\mathbb{C} \downarrow B)$  which lies in  $\mathbf{Spl}_\Gamma(E, p)$  for *some* monadic extension  $(E, p)$ . The full subcategory of  $(\mathbb{C} \downarrow B)$

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Dedicated to George Janelidze on the occasion of his sixtieth birthday.

<sup>1</sup>The right adjoint  $H: \mathbb{X} \rightarrow \mathbb{C}$  is usually not required to be fully faithful, in which case it is necessary to consider moreover a class  $\mathcal{F}$  of morphisms in  $\mathbb{X}$ . Here we make the same restriction as in [24].

of all coverings of  $B$  is denoted by  $\text{Cov}_\Gamma(B)$ , and the full subcategory of the arrow category  $\text{Arr}(\mathbb{C})$  containing the coverings of *every*  $B \in \mathbb{C}$  by  $\text{Cov}_\Gamma(\mathbb{C})$ . Finally, a *normal extension* is a monadic extension  $(E, p)$  which lies in  $\text{Spl}_\Gamma(E, p)$ .

We are particularly interested in the following example of a Galois structure  $\Gamma = (\mathbb{C}, \mathbb{X}, H, I, \eta, \mathcal{E})$  [19]:  $\mathbb{C}$  is the variety of groups,  $\mathbb{X}$  is the variety of abelian groups,  $H: \mathbb{X} \rightarrow \mathbb{C}$  is the inclusion functor hence  $I: \mathbb{C} \rightarrow \mathbb{X}$  is the abelianisation functor and  $\eta_A$  is the quotient map  $A \rightarrow A/[A, A]$ , and  $\mathcal{E}$  is the class of surjective homomorphisms. In this case, every  $p \in \mathcal{E}$  is a monadic extension. A trivial covering is a surjective homomorphism  $f: A \rightarrow B$  whose restriction to the commutator subgroups  $[A, A] \rightarrow [B, B]$  is an isomorphism. The coverings are precisely the central extensions: surjective homomorphisms  $f: A \rightarrow B$  whose kernel  $\text{Ker}(f)$  is contained in the center of  $A$ . Normal extensions and coverings coincide in this case. We shall denote the full subcategory of the arrow category  $\text{Arr}(\mathbb{C})$  containing all surjective homomorphisms (=extensions) by  $\text{Ext}(\mathbb{C})$ .

Of particular interest to us here is that the inclusion functor  $H_1: \text{Cov}_\Gamma(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{C})$  has a left adjoint  $I_1: \text{Ext}(\mathbb{C}) \rightarrow \text{Cov}_\Gamma(\mathbb{C})$ ; its value at a surjective homomorphism  $f: A \rightarrow B$  is given by the induced homomorphism  $A/[\text{Ker}(f), A] \rightarrow B$  and the  $f$ -component of the reflection unit,  $\eta_f^1: f \rightarrow H_1 I_1(f)$ , by the quotient map  $A \rightarrow A/[\text{Ker}(f), A]$ . This fact allowed G. Janelidze to introduce a notion of *double central extension* of groups in [20]. These are defined as the coverings with respect to the Galois structure  $\Gamma_1 = (\mathbb{C}_1, \mathbb{X}_1, H_1, I_1, \eta^1, \mathcal{E}^1)$ , where  $\mathbb{C}_1 = \text{Ext}(\mathbb{C})$ ,  $\mathbb{X}_1 = \text{Cov}_\Gamma(\mathbb{C})$ ,  $H_1$ ,  $I_1$  and  $\eta^1$  are as above, and  $\mathcal{E}^1$  is a suitably defined class of morphisms in  $\text{Ext}(\mathbb{C})$ .

For an arbitrary Galois structure  $\Gamma = (\mathbb{C}, \mathbb{X}, H, I, \eta, \mathcal{E})$  it is not always true that every morphism  $f \in \mathcal{E}$  admits a reflection in  $\text{Cov}_\Gamma(\mathbb{C})$ , however this is the case for many important examples. A comprehensive study of conditions on  $\Gamma$  under which “coverings are reflective” can be found in [24]. For the example of groups considered here, we have an even stronger property: not only do we have a left adjoint for  $H_1: \text{Cov}_\Gamma(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{C})$ , which gives rise to the Galois structure  $\Gamma_1$  and a notion of double central extension, but now also the inclusion functor  $H_2: \text{Cov}_{\Gamma_1}(\mathbb{C}_1) \rightarrow \text{Ext}(\mathbb{C}_1)$  admits a left adjoint. (Here we have written  $\text{Ext}(\mathbb{C}_1)$  for the full subcategory of  $\text{Arr}(\mathbb{C}_1)$  determined by the class  $\mathcal{E}^1$ .) This, in its turn, yields yet another Galois structure  $\Gamma_2 = (\mathbb{C}_2, \mathbb{X}_2, H_2, I_2, \eta^2, \mathcal{E}^2)$  with  $\mathbb{C}_2 = \text{Ext}(\mathbb{C}_1)$ ,  $\mathbb{X}_2 = \text{Cov}_{\Gamma_1}(\mathbb{C}_1)$  and  $\mathcal{E}^2$  a suitably defined class of morphisms in  $\text{Ext}(\mathbb{C}_1)$ , which now allows us to define *triple central extensions*, and so on. More specifically, as was proved by G. Janelidze and presented in his talk [21], we have an infinite chain of Galois structures  $\Gamma_n = (\mathbb{C}_n, \mathbb{X}_n, H_n, I_n, \eta^n, \mathcal{E}^n)$  such that for each  $n \geq 1$ ,  $\mathbb{C}_n$  is the full subcategory of  $\text{Arr}(\mathbb{C}_{n-1})$  determined by  $\mathcal{E}^{n-1}$ , and  $\mathbb{X}_n$  consists exactly of the coverings with respect to  $\Gamma_{n-1}$ . (Here we assumed that  $\mathbb{C}_0 = \mathbb{C}$ ,  $\mathcal{E}^0 = \mathcal{E}$  and  $\Gamma_0 = \Gamma$ .) Let us also mention here that, just as for the case  $n = 0$ , we have for each  $n \geq 1$  that every morphism in  $\mathcal{E}^n$  is a monadic extension, and that every covering with respect to  $\Gamma_n$  is a normal extension.

The particular Galois structure  $\Gamma$  considered above is, in fact, the prototypical example of a large class of Galois structures studied in [23]: namely, those  $\Gamma = (\mathbb{C}, \mathbb{X}, H, I, \eta, \mathcal{E})$  which are *admissible* in the sense explained below, for which  $\mathbb{C}$  is an exact category in the sense of [1],  $H: \mathbb{X} \rightarrow \mathbb{C}$  is the inclusion functor into  $\mathbb{C}$  of a Birkhoff subcategory  $\mathbb{X}$  (=a full reflective subcategory closed under subobjects and regular quotient objects) and  $\mathcal{E}$  is the class of regular epimorphisms in  $\mathbb{C}$ . The

exactness of  $\mathbb{C}$  implies, in particular, that every regular epimorphism is a monadic extension. If  $\mathbb{C}$  is, moreover, *Mal'tsev* [8, 9] (=the permutability condition  $RS = SR$  is satisfied for every pair of internal equivalence relations  $R$  and  $S$  on an object of  $\mathbb{C}$ ) then every covering is a normal extension (see [23], where the assumption on  $\mathbb{C}$  was actually slightly weaker). In this case we moreover have that every Birkhoff subcategory of  $\mathbb{C}$  is admissible.

The coverings with respect to such a  $\Gamma$  were called *central extensions* in [23]. This choice of terminology is motivated not only by the example of groups considered above, but also by the fact that when  $\mathbb{C}$  is any variety of  $\Omega$ -groups then the coverings with respect to  $\Gamma$  are precisely the central extensions studied by A. Fröhlich and others, defined with respect to the subvariety  $\mathbb{X}$ . Also, when  $\mathbb{C}$  is, more generally, a Mal'tsev variety and  $\mathbb{X}$  is the subvariety of abelian algebras in  $\mathbb{C}$ , then the coverings with respect to  $\Gamma = (\mathbb{C}, \mathbb{X}, H, I, \eta, \mathcal{E})$  are precisely the central extensions arising from commutator theory in universal algebra: namely, those surjective homomorphisms  $f: A \rightarrow B$  for which the commutator  $[A \times_B A, A \times_B A]$  of the kernel congruence  $A \times_B A$  of  $f$  with largest congruence  $A \times A$  on  $A$  is trivial (see [25]).

Now let us turn our attention back to the infinite chain of Galois structures  $\Gamma_n$  ( $n \geq 0$ ) described above. Its existence is not specific to that particular  $\Gamma$ : in [13] it was shown that whenever  $\mathbb{C}$  is a semi-abelian category and  $\mathbb{X}$  is a Birkhoff subcategory of  $\mathbb{C}$ , then the corresponding Galois structure  $\Gamma = (\mathbb{C}, \mathbb{X}, H, I, \eta, \mathcal{E})$  (where  $\mathcal{E}$  is the class of regular epimorphisms in  $\mathbb{C}$ ) induces a similar chain of Galois structures  $\Gamma_n = (\mathbb{C}_n, \mathbb{X}_n, H_n, I_n, \eta^n, \mathcal{E}^n)$  ( $n \geq 1$ ), i.e. we have for each  $n \geq 1$  that  $\mathbb{C}_n$  is the full subcategory of  $\text{Arr}(\mathbb{C}_{n-1})$  determined by  $\mathcal{E}^{n-1}$ , and  $\mathbb{X}_n$  consists exactly of the coverings with respect to  $\Gamma_{n-1}$ . Note that since any such  $\Gamma$  is of the type studied in [23], it makes sense to call the coverings with respect to  $\Gamma_{n-1}$  *n-fold central extensions*, as in the case of groups.

The aim of the present article is to further extend the latter result by showing the existence of such a chain of “higher order” Galois structures under conditions on the Galois structure  $\Gamma = (\mathbb{C}, \mathbb{X}, H, I, \eta, \mathcal{E})$  which are in particular satisfied when  $\mathbb{C}$  is exact Mal'tsev,  $\mathbb{X}$  is a Birkhoff subcategory of  $\mathbb{C}$  with inclusion functor  $H: \mathbb{X} \rightarrow \mathbb{C}$  and  $\mathcal{E}$  is the class of regular epimorphisms in  $\mathbb{C}$ . This includes the situation considered in [13], but we retrieve also the notion of double central extension considered in [17] and [14] in the exact Mal'tsev context, and extend it to higher dimensions. To obtain our result, it will be sufficient to show the existence of the Galois structure  $\Gamma_1 = (\mathbb{C}_1, \mathbb{X}_1, H_1, I_1, \eta^1, \mathcal{E}^1)$  and to prove that it satisfies the same conditions as imposed on  $\Gamma$ . The existence of the  $\Gamma_n$  ( $n \geq 2$ ) will then simply follow by induction. Under our assumptions we shall moreover have for each  $n \geq 1$  that every morphism in  $\mathcal{E}^n$  is a monadic extension, and that every covering with respect to  $\Gamma_n$  (i.e. every  $(n+1)$ -fold central extension) is a normal extension.

To conclude this introduction, let us remark that the importance of higher dimensional central extensions lies foremost in their use in non-abelian homological algebra, and in particular in their connection to the Brown-Ellis-Hopf formulae for the homology of a group [7]. For more details, see for instance [10, 12, 13, 15, 22, 31].

## 2. REFLECTIVENESS OF COVERINGS

*Throughout this section, we will fix a Galois structure  $\Gamma = (\mathbb{C}, \mathbb{X}, H, I, \eta, \mathcal{E})$ .*

We use the notation  $\text{Ext}(\mathbb{C})$  for the full subcategory of the arrow category  $\text{Arr}(\mathbb{C})$  whose objects are the morphisms in  $\mathcal{E}$ . Note, however, that we do not, initially, require every  $f \in \mathcal{E}$  to be a monadic extension.

We are interested in conditions on  $\Gamma$  under which a left adjoint exists for the inclusion functor  $\text{Cov}_\Gamma(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{C})$ . A property which plays an important role in this is *admissibility*, hence we begin by recalling what it means.

As before, we use the notation  $(\mathbb{C} \downarrow B)$  for the full subcategory of the slice category  $\mathbb{C}/B$  containing those  $f \in \mathbb{C}/B$  which lie in  $\mathcal{E}$ . Also, we write  $(\mathbb{X} \downarrow Y)$  for the full subcategory of the slice category  $\mathbb{X}/Y$  containing those  $(X, \varphi) \in \mathbb{X}/Y$  for which  $H(\varphi) \in \mathcal{E}$ . Since  $H(I(\mathcal{E})) \subseteq \mathcal{E}$ , the reflector  $I: \mathbb{C} \rightarrow \mathbb{X}$  extends, for any  $B \in \mathbb{C}$ , to a functor  $I^B: (\mathbb{C} \downarrow B) \rightarrow (\mathbb{X} \downarrow I(B))$  in an obvious way.  $I^B$  has a right adjoint  $H^B: (\mathbb{X} \downarrow I(B)) \rightarrow (\mathbb{C} \downarrow B)$ , sending an object  $(X, \varphi) \in (\mathbb{X} \downarrow I(B))$  to  $(A, f) \in (\mathbb{C} \downarrow B)$  defined via the pullback

$$\begin{array}{ccc} A & \longrightarrow & H(X) \\ f \downarrow & & \downarrow H(\varphi) \\ B & \xrightarrow{\eta_B} & HI(B). \end{array}$$

**Definition 2.1.** The Galois structure  $\Gamma$  is called *admissible* when the functors  $H^B: (\mathbb{X} \downarrow I(B)) \rightarrow (\mathbb{C} \downarrow B)$  are fully faithful.

Let us also recall the following

**Lemma 2.2.** [24, Proposition 2.4] *If  $\Gamma$  is admissible then  $I: \mathbb{C} \rightarrow \mathbb{X}$  preserves those pullbacks*

$$(A) \quad \begin{array}{ccc} D & \longrightarrow & A \\ h \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

for which  $f$  is a trivial covering.

Consequently, if  $f$  in the pullback diagram (A) is a trivial covering, then so is  $h$ . And, if we assume that every pullback of a monadic extension is a monadic extension (which is certainly the case if every morphism in the pullback-stable class  $\mathcal{E}$  is a monadic extension, as we shall assume later on) then we moreover have the following: if, once again in the pullback diagram (A),  $f$  is split by a monadic extension  $p: E \rightarrow B$ , then  $h$  is split by  $g^*(E, p)$ . Hence:

**Lemma 2.3.** *If  $\Gamma$  is admissible and monadic extensions are pullback-stable, then coverings are pullback-stable as well.*

We return to our problem of finding a left adjoint for the inclusion functor  $H_1: \text{Cov}_\Gamma(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{C})$ . If it exists, then it will necessarily restrict, for every  $B$ , to a left adjoint for the inclusion functor  $\text{Cov}_\Gamma(B) \rightarrow (\mathbb{C} \downarrow B)$ . This is a consequence of the fact that every identity morphism is a covering and  $\text{Cov}_\Gamma(\mathbb{C})$  is a replete subcategory of  $\text{Ext}(\mathbb{C})$  (see Corollary 5.2 in [18]). Moreover, if we assume that  $\Gamma$  is admissible and that monadic extensions are pullback-stable, so that coverings are pullback-stable, then the existence of the latter left adjoints is also sufficient for  $H_1$  to admit a left adjoint (by Proposition 5.8 in [18]). Thus, we

shall be looking for conditions on  $\Gamma$  under which, for every  $B \in \mathbb{C}$ , the inclusion functor  $\text{Cov}_\Gamma(B) \longrightarrow (\mathbb{C} \downarrow B)$  admits a left adjoint.

As remarked in [24], it often happens that there exists, for an object  $B$ , a single monadic extension  $p: E \longrightarrow B$  which splits every covering of  $B$ , i.e. such that  $\text{Cov}(B) = \text{Spl}(E, p)$ . For instance,  $p: E \longrightarrow B$  may be such that  $E$  is projective with respect to all monadic extensions. Clearly, in this case, it will suffice for us to find a left adjoint for the inclusion functor  $\text{Spl}(E, p) \longrightarrow (\mathbb{C} \downarrow B)$ , for every monadic extension  $p: E \longrightarrow B$ .

However, here it will be unnecessary to assume the existence of such monadic extensions  $p: E \longrightarrow B$  for which  $\text{Cov}(B) = \text{Spl}(E, p)$ , since the existence of left adjoints for the inclusion functors  $\text{Spl}(E, p) \longrightarrow (\mathbb{C} \downarrow B)$  automatically implies that of left adjoints for the inclusion functors  $\text{Cov}_\Gamma(B) \longrightarrow (\mathbb{C} \downarrow B)$  in the particular situation we are interested in. Indeed, as already mentioned, we will be interested in cases where every  $p \in \mathcal{E}$  is a monadic extension, and every covering is a normal extension. Under these assumptions, and if moreover  $\Gamma$  is admissible, any reflection  $(\bar{E}, \bar{p})$  in  $\text{Spl}(E, p)$  of an object  $(E, p) \in (\mathbb{C} \downarrow B)$  is necessarily also the reflection in  $\text{Cov}(B)$  of  $(E, p)$ . Indeed, for every morphism  $h: (E, p) \longrightarrow (A, f)$  to a covering  $f: A \longrightarrow B$  of  $B$  there is a unique morphism  $(\bar{E}, \bar{p}) \longrightarrow (A, f)$  rendering the diagram

$$\begin{array}{ccc} (E, p) & \xrightarrow{\quad} & (\bar{E}, \bar{p}) \\ & \searrow h & \swarrow \text{dotted} \\ & (A, f) & \end{array}$$

commutative because the existence of the morphism  $h: (E, p) \longrightarrow (A, f)$ , i.e. of the commutative triangle

$$\begin{array}{ccc} E & \xrightarrow{h} & A \\ & \searrow p & \swarrow f \\ & B & \end{array}$$

forces  $(A, f)$  to be split by  $(E, p)$ : since  $f: A \longrightarrow B$  is a covering, hence a normal extension, by assumption, we have that  $f^*(A, f)$  is a trivial covering. Hence, the pullback-stability of trivial coverings (by Lemma 2.2) tells us that also  $h^*f^*(A, f) \cong (fh)^*(A, f) = p^*(A, f)$  is a trivial covering, i.e.  $(A, f) \in \text{Spl}(E, p)$ .

We have just proved the following lemma:

**Lemma 2.4.** *Assume that  $\Gamma$  is admissible, that every morphism in  $\mathcal{E}$  is a monadic extension and that every covering is a normal extension. Then  $H_1: \text{Cov}_\Gamma(\mathbb{C}) \longrightarrow \text{Ext}(\mathbb{C})$  admits a left adjoint as soon as for every  $p: E \longrightarrow B \in \mathcal{E}$  the inclusion functor  $\text{Spl}(E, p) \longrightarrow (\mathbb{C} \downarrow B)$  admits a left adjoint.*

In [24] various sufficient conditions were given for left adjoints for the inclusion functors  $\text{Spl}(E, p) \longrightarrow (\mathbb{C} \downarrow B)$  to exist, covering many important examples. Here we shall be interested only in the following one: the preservation by the functor  $I: \mathbb{C} \longrightarrow \mathbb{X}$  of those pullbacks  $(\mathbf{A})$  for which  $f, g \in \mathcal{E}$  and  $g$  is a split epimorphism. Note that this condition is slightly stronger than what was required in Proposition 5.2(b) in [24]. In the “absolute” case, where  $\mathcal{E}$  consists of all regular epimorphisms in  $\mathbb{C}$ , this stronger property was first considered in relationship with central extensions in [16]. In [6] the authors say in this case that the reflector  $I: \mathbb{C} \longrightarrow \mathbb{X}$  is *regular* if moreover every reflection unit  $\eta_A: A \longrightarrow HI(A)$  is a regular epimorphism.

**Lemma 2.5.** [24, Proposition 5.2(b)] *If  $\Gamma$  is admissible, and  $I: \mathbb{C} \rightarrow \mathbb{X}$  preserves those pullbacks **(A)** for which  $f, g \in \mathcal{E}$  and  $g$  is a split epimorphism, then the inclusion functor  $\text{Spl}(E, p) \rightarrow (\mathbb{C} \downarrow B)$  admits a left adjoint, for every monadic extension  $p: E \rightarrow B$ .*

Hence, under the assumptions of Lemma's 2.4 and 2.5, the inclusion functor  $H_1: \text{Cov}_\Gamma(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{C})$  has a left adjoint. Moreover, the condition that every covering is a normal extension follows from the other assumptions. Indeed, first of all we have:

**Lemma 2.6.** *If  $I: \mathbb{C} \rightarrow \mathbb{X}$  preserves those pullbacks **(A)** for which  $f, g \in \mathcal{E}$  and  $g$  is a split epimorphism, then every covering which is a split epimorphism is a trivial covering.*

*Proof.* The proof of Proposition 4.5(2) in [13] remains valid. However, one should replace  $\Gamma_n$  by the Galois structure  $\Gamma$  considered here and, in particular, “higher order extensions” by morphism in  $\mathcal{E}$ . To avoid confusion, we recall the argument.

Let  $f: A \rightarrow B$  be a covering which is also a split epimorphism. Then there exists a diagram

$$\begin{array}{ccccc} P & \longrightarrow & A & \xrightarrow{\eta_A} & HI(A) \\ g \downarrow & & \downarrow f & & \downarrow HI(f) \\ E & \xrightarrow{p} & B & \xrightarrow{\eta_B} & HI(B) \end{array}$$

in which the left hand square is a pullback,  $p$  is a monadic extension and  $g$  a trivial covering. We must prove that the right hand square too is a pullback. Let us write  $Q = B \times_{HI(B)} HI(A)$  for the pullback of  $\eta_B$  and  $HI(f)$ . To see that the canonical morphism  $A \rightarrow Q$  is an isomorphism, we consider it as a morphism  $(A, f) \rightarrow (Q, \eta_B^*(HI(f)))$  in  $(\mathbb{C} \downarrow B)$  and note that its image by the functor  $p^*: (\mathbb{C} \downarrow B) \rightarrow (\mathbb{C} \downarrow E)$  is an isomorphism. The latter is the case since in the diagram above the exterior rectangle is a pullback. Indeed, it coincides with the exterior rectangle in the diagram

$$\begin{array}{ccccc} P & \xrightarrow{\eta_P} & HI(P) & \longrightarrow & HI(A) \\ g \downarrow & & \downarrow HI(g) & & \downarrow HI(f) \\ E & \xrightarrow{\eta_E} & HI(E) & \xrightarrow{HI(p)} & HI(B) \end{array}$$

which is a pullback since both of its squares are: the left hand one because  $g$  is a trivial covering, and the right hand square since it is the image by  $HI$  of the left hand square in the previous diagram, which is a pullback preserved by  $I$  (hence also by  $HI$ ) by assumption.

It suffices now to observe that  $p^*: (\mathbb{C} \downarrow B) \rightarrow (\mathbb{C} \downarrow E)$  reflects isomorphisms since it is monadic.  $\square$

If we moreover have that coverings are pullback-stable, as is certainly the case when  $\Gamma$  is admissible and every morphism in  $\mathcal{E}$  is a monadic extension (by Lemma 2.3) then the kernel pair projections  $A \times_B A \rightarrow A$  of any covering  $A \rightarrow B$  are coverings as well and therefore trivial coverings by the previous lemma. Whence:

**Lemma 2.7.** *Assume that  $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \mathcal{E})$  is admissible, that every morphism in the class  $\mathcal{E}$  is a monadic extension, and that  $I: \mathbb{C} \rightarrow \mathbb{X}$  preserves those pullbacks **(A)** for which  $f, g \in \mathcal{E}$  and  $g$  is a split epimorphism. Then every covering is a normal extension.*

Finally, Lemma's 2.4, 2.5 and 2.7 yield

**Theorem 2.8.** *Assume that  $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \mathcal{E})$  is admissible, that every morphism in the class  $\mathcal{E}$  is a monadic extension, and that  $I: \mathbb{C} \rightarrow \mathbb{X}$  preserves those pullbacks **(A)** for which  $f, g \in \mathcal{E}$  and  $g$  is a split epimorphism. Then the inclusion functor  $\text{Cov}_\Gamma(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{C})$  admits a left adjoint and, moreover, every covering is a normal extension.*

**2.1. Proof of Lemma 2.5.** Before going to the next section, it is useful to give a proof for Lemma 2.5, as we shall not only be needing the existence, but also the construction of the left adjoints of the inclusion functors  $\text{Cov}_\Gamma(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{C})$ , later on.

For this, note first of all that when  $\Gamma$  is admissible, i.e. if for every  $B$ , the right adjoint  $H^B$  in the adjunction

$$(\mathbb{C} \downarrow B) \begin{array}{c} \xrightarrow{I^B} \\ \xleftarrow{H^B} \end{array} (\mathbb{X} \downarrow I(B))$$

is fully faithful, then the (essential) image of any functor  $H^B$  consists exactly of the trivial coverings of  $B$ , and the composite  $H^B I^B$  provides a left adjoint for the inclusion functor  $\text{Triv}_\Gamma(B) \rightarrow (\mathbb{C} \downarrow B)$ . (Here we have written  $\text{Triv}_\Gamma(B)$  for the full subcategory  $(\mathbb{C} \downarrow B)$  consisting of all trivial coverings.) Hence we have that “trivial coverings are reflective”. In order to deduce from this the reflectiveness of coverings or, more specifically, the existence of left adjoints for the inclusion functors  $\text{Spl}(E, p) \rightarrow (\mathbb{C} \downarrow B)$ , we need to say a few words about what it means for a morphism to be a monadic extension.

First of all we note that a left adjoint for the pullback-functor  $p^*: (\mathbb{C} \downarrow B) \rightarrow (\mathbb{C} \downarrow E)$  exists for *any* morphism  $p: E \rightarrow B$  in  $\mathcal{E}$ , and is given by composition with  $p$ . We denote it  $\Sigma_p$  and write  $T^p = p^* \Sigma_p$  for the corresponding monad on  $(\mathbb{C} \downarrow E)$ . Next, we recall that a (downward) morphism

$$\begin{array}{ccc} S & \rightrightarrows & C \\ h \downarrow & & \downarrow g \\ R & \rightrightarrows & E \end{array}$$

of internal equivalence relations is a *discrete fibration* if the commutative square involving the second projections (hence, also the square involving the first projections) is a pullback. For a fixed equivalence relation  $R$ , we write  $\mathbb{C}^R$  for the category of discrete fibrations  $(g, h): S \rightarrow R$ , with the obvious morphisms, and  $\mathbb{C}^{\downarrow R}$  (respectively,  $\mathbb{C}^{\downarrow_\iota R}$ ) for the full subcategory consisting of those  $(g, h): S \rightarrow R$  for which  $g$  and  $h$  lie in  $\mathcal{E}$  (respectively, are trivial coverings).

Now, let us fix a morphism  $p: E \rightarrow B$  in  $\mathcal{E}$  and write  $Eq(p)$  for the equivalence relation  $E \times_B E \rightrightarrows E$  (i.e. the kernel pair of  $p$ ). By sending any morphism  $f: A \rightarrow B$  in  $\mathcal{E}$  to the discrete fibration displayed in the left hand side of the

diagram

$$(B) \quad \begin{array}{ccccc} E \times_B E \times_B A & \rightrightarrows & E \times_B A & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow f \\ E \times_B E & \rightrightarrows & E & \xrightarrow{p} & B \end{array}$$

obtained by first pulling back  $f$  along  $p$ , and next taking kernel pairs, we obtain a functor  $K^p: (\mathbb{C} \downarrow B) \rightarrow \mathbb{C}^{\downarrow Eq(p)}$ . It turns out (see, for instance, [28, 29]) that  $\mathbb{C}^{\downarrow Eq(p)}$  is equivalent to the category  $(\mathbb{C} \downarrow E)^{T^p}$  of (Eilenberg-Moore) algebras for the monad  $T^p$ , and that  $K^p: (\mathbb{C} \downarrow B) \rightarrow \mathbb{C}^{\downarrow Eq(p)}$  corresponds, via this equivalence, to the comparison functor  $K^{T^p}: (\mathbb{C} \downarrow B) \rightarrow (\mathbb{C} \downarrow E)^{T^p}$ , whence:

**Lemma 2.9.** [28, 29] *A morphism  $p: E \rightarrow B$  in  $\mathcal{E}$  is a monadic extension if, and only if, the functor  $K^p: (\mathbb{C} \downarrow B) \rightarrow \mathbb{C}^{\downarrow Eq(p)}$  is an equivalence of categories.*

Suppose now that the conditions of Lemma 2.5 are satisfied, and let  $p: E \rightarrow B$  be a monadic extension. Then, since trivial coverings are pullback-stable (by Lemma 2.2) and by definition of  $\mathbf{Spl}(E, p)$ , the equivalence  $K^p: (\mathbb{C} \downarrow B) \rightarrow \mathbb{C}^{\downarrow Eq(p)}$  restricts to an equivalence  $\mathbf{Spl}(E, p) \rightarrow \mathbb{C}^{\downarrow \iota Eq(p)}$ . Therefore, in order to find a left adjoint for the inclusion functor  $\mathbf{Spl}(E, p) \rightarrow (\mathbb{C} \downarrow B)$  it suffices to find one for the inclusion functor  $\mathbb{C}^{\downarrow \iota Eq(p)} \rightarrow \mathbb{C}^{\downarrow Eq(p)}$ . But we already know from above that “trivial coverings are reflective” so that the inclusion functors  $\mathbf{Triv}_\Gamma(E) \rightarrow (\mathbb{C} \downarrow E)$  and  $\mathbf{Triv}_\Gamma(E \times_B E) \rightarrow (\mathbb{C} \downarrow E \times_B E)$  have left adjoints  $H^E I^E: (\mathbb{C} \downarrow E) \rightarrow \mathbf{Triv}_\Gamma(E)$  and  $H^{E \times_B E} I^{E \times_B E}: (\mathbb{C} \downarrow E \times_B E) \rightarrow \mathbf{Triv}_\Gamma(E \times_B E)$ , respectively. And, we have that these extend to a left adjoint for  $\mathbb{C}^{\downarrow \iota Eq(p)} \rightarrow \mathbb{C}^{\downarrow Eq(p)}$ , as a consequence of the assumption on  $I: \mathbb{C} \rightarrow \mathbb{X}$ . To see this, consider a discrete fibration  $(g, h): S \rightarrow Eq(p)$  as depicted below. Applying to it the functor  $HI: \mathbb{C} \rightarrow \mathbb{C}$  and then pulling back along the units  $\eta_E: E \rightarrow HI(E)$  and  $\eta_{E \times_B E}: E \times_B E \rightarrow HI(E \times_B E)$  yields the right hand diagram

$$\begin{array}{ccc} S & \xrightleftharpoons[\pi_2]{\pi_1} & C \\ \downarrow h & & \downarrow g \\ E \times_B E & \rightrightarrows & E \end{array} \quad \begin{array}{ccc} \bar{S} & \xrightleftharpoons[\bar{\pi}_2]{\bar{\pi}_1} & \bar{C} \\ \downarrow \bar{h} & & \downarrow \bar{g} \\ E \times_B E & \rightrightarrows & E \end{array}$$

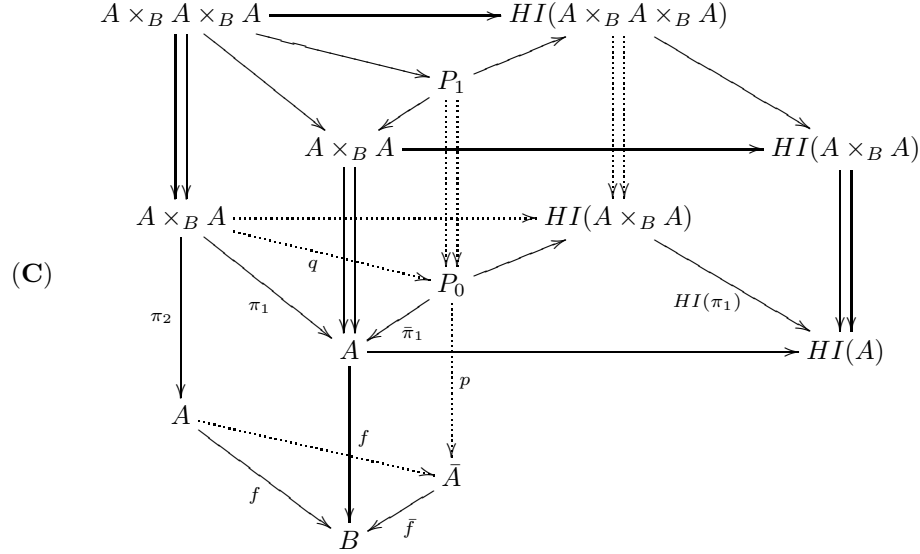
where  $\bar{g} = H^E I^E(g)$  and  $\bar{h} = H^{E \times_B E} I^{E \times_B E}(h)$ . We claim that it is a discrete fibration of equivalence relations, hence an object of  $\mathbb{C}^{\downarrow \iota Eq(p)}$ . Since, in the left hand diagram, the commutative square involving the second projections is a pullback which is preserved by  $I$  (hence also by  $HI$ ) by assumption, we clearly have that the corresponding commutative square in the right hand diagram is a pullback too. From this it follows easily that  $\bar{S}$  is a relation on  $\bar{C}$ , which is both reflexive and symmetric. It is moreover transitive, since the pullback

$$\begin{array}{ccc} S \times_C S & \longrightarrow & S \\ \downarrow & & \downarrow \pi_1 \\ S & \xrightarrow{\pi_2} & C \end{array}$$



is preserved by  $I$  (hence by  $HI$ ), by assumption. Thus we obtain a functor  $\mathbb{C}^{\downarrow Eq(p)} \rightarrow \mathbb{C}^{\downarrow_t Eq(p)}$ , which is clearly left adjoint to  $\mathbb{C}^{\downarrow_t Eq(p)} \rightarrow \mathbb{C}^{\downarrow Eq(p)}$ . This concludes the proof of Lemma 2.5.

**2.2. Construction of the left adjoint for  $H_1: \mathbb{C}Ext_{\Gamma}(\mathbb{C}) \rightarrow Ext(\mathbb{C})$ .** We have just described the left adjoints of the inclusion functors  $Spl(E, p) \rightarrow (\mathbb{C} \downarrow B)$ . Tracing back our steps, we thus find a left adjoint for  $H_1: Cov_{\Gamma}(\mathbb{C}) \rightarrow Ext(\mathbb{C})$ , sending a morphism  $f: A \rightarrow B$  in  $\mathcal{E}$  to the covering  $\bar{f}: \bar{A} \rightarrow B$  obtained as follows (as depicted in the diagram below): first we pull  $f: A \rightarrow B$  back along itself, and then we take kernel pairs. This yields the discrete fibration  $K^f(A, f)$  to which we apply the functor  $HI: \mathbb{C} \rightarrow \mathbb{C}$ . Next we pull back along the units  $\eta_A: A \rightarrow HI(A)$  and  $\eta_{A \times_B A}: A \times_B A \rightarrow HI(A \times_B A)$  and obtain a discrete fibration in  $\mathbb{C}^{\downarrow_t Eq(f)}$ . The covering in  $Spl(A, f)$  which corresponds to this discrete fibration via the equivalence  $Spl(A, f) \cong \mathbb{C}^{\downarrow_t Eq(f)}$  is the desired  $\bar{f}$ .



### 3. HIGHER CENTRAL EXTENSIONS

As before, we fix a Galois structure  $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \mathcal{E})$ .

We now wish to consider conditions on  $\Gamma$ , as well as define a class  $\mathcal{E}^1$  of morphisms in  $Ext(\mathbb{C})$ , such that

- (1) the conditions on  $\Gamma$  imply the assumptions of Theorem 2.8. Hence, in particular, the inclusion functor  $H_1: Cov_{\Gamma}(\mathbb{C}) \rightarrow Ext(\mathbb{C})$  admits a left adjoint  $I_1: Ext(\mathbb{C}) \rightarrow Cov_{\Gamma}(\mathbb{C})$  with reflection unit  $\eta^1$ .
- (2)  $\Gamma_1 = (\mathbb{C}_1, \mathbb{X}, I_1, H_1, \eta^1, \mathcal{E}^1)$  is a Galois structure satisfying the same conditions as  $\Gamma$ .

Hence, when  $\Gamma$  satisfies these conditions, we will obtain, inductively, for each  $n \geq 1$  a Galois structure  $\Gamma_n = (\mathbb{C}_n, \mathbb{X}_n, H_n, I_n, \eta^n, \mathcal{E}^n)$  such that  $\mathbb{C}_n = Ext(\mathbb{C}_{n-1})$ ,  $\mathcal{E}^n = (\mathcal{E}^{n-1})^1$  and  $\mathbb{X}_n$  consists exactly of the coverings with respect to  $\Gamma_{n-1}$ .

We begin by defining the class  $\mathcal{E}^1$  of morphisms in  $\text{Ext}(\mathbb{C})$ . If  $a, b \in \mathcal{E}$  then a morphism  $a \longrightarrow b$  in  $\text{Ext}(\mathbb{C})$  is a commutative square

$$(D) \quad \begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

in  $\mathbb{C}$ . Certainly, we want every morphism in  $\mathcal{E}^1$  to admit pullbacks along arbitrary morphisms in  $\text{Ext}(\mathbb{C})$ . Since we want  $\mathcal{E}^1$  (respectively,  $\Gamma_1$ ) to inherit the properties of  $\mathcal{E}$  (respectively,  $\Gamma$ ) it is natural to require moreover that such pullbacks be pointwise—the inclusion functor  $\text{Ext}(\mathbb{C}) \longrightarrow \text{Arr}(\mathbb{C})$  preserves them—and that every morphism  $(E)$  in  $\mathcal{E}^1$  has its “horizontal” arrows  $f$  and  $f'$  too in  $\mathcal{E}$ . We define  $\mathcal{E}^1$  as the class of morphisms in  $\text{Ext}(\mathbb{C})$  satisfying these properties.

**Proposition 3.1.** *The above defined class  $\mathcal{E}^1$  consists exactly of those commutative squares*

$$(E) \quad \begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

of morphisms in  $\mathcal{E}$  for which also the induced morphism  $(a, f'): A' \longrightarrow A \times_B B'$  to the pullback of  $b$  and  $f$  lies in  $\mathcal{E}$ . Hence, the class  $\mathcal{E}^1$  is the same  $\mathcal{E}^1$  as considered for instance in [13] in a semi-abelian context.

*Proof.* Let us write  $\mathcal{E}'$  for the class of morphisms in  $\text{Ext}(\mathbb{C})$  just defined. We must prove that  $\mathcal{E}' = \mathcal{E}^1$ .

The validity of the inclusion  $\mathcal{E}' \subseteq \mathcal{E}^1$  is straightforward and has been observed many times before with varying assumptions on  $\mathbb{C}$  and  $\mathcal{E}$ —see, for instance, Proposition 3.5(1) in [13] for a proof. For  $\mathcal{E}' \supseteq \mathcal{E}^1$  it suffices to note that for each commutative square  $(E)$  in  $\mathcal{E}^1$ , the induced morphism  $A' \longrightarrow P$  to the pullback  $P = A \times_B B'$  of  $b$  and  $f$  is the pullback in  $\text{Arr}(\mathbb{C})$  of the morphisms  $(f', f): a \longrightarrow b$  and  $(1_{B'}, b): 1_{B'} \longrightarrow b$ :

□

Let us then consider the following list of conditions on  $\Gamma$ , which we do not assume to be admissible, a priori.

- (E4) If  $g \circ f \in \mathcal{E}$  then  $g \in \mathcal{E}$ .  
 (E5) Any split epimorphism in  $\text{Ext}(\mathbb{C})$ , i.e. any diagram

$$\begin{array}{ccc} A' & \xrightleftharpoons[f']{s'} & B' \\ a \downarrow & & \downarrow b \\ A & \xrightleftharpoons[f]{s} & B \end{array}$$

- in  $\mathbb{C}$  for which  $a, b \in \mathcal{E}$  and  $f' \circ s' = 1_{B'}$ ,  $f \circ s = 1_B$ ,  $b \circ f' = f \circ a$  and  $a \circ s' = s \circ b$ , lies in  $\mathcal{E}^1$ .  
 (M) Every morphism in  $\mathcal{E}$  is a monadic extension.  
 (B)  $\mathbb{X}$  is a *strongly  $\mathcal{E}$ -Birkhoff* subcategory of  $\mathbb{C}$ : for every  $f: A \rightarrow B \in \mathcal{E}$ , the induced commutative square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ f \downarrow & & \downarrow HI(f) \\ B & \xrightarrow{\eta_B} & HI(B) \end{array} \quad (\mathbf{F})$$

is in  $\mathcal{E}^1$ .

Let  $\mathbb{C}$  be a category with finite limits and coequalisers, and let  $\mathcal{E}$  consist of all regular epimorphisms in  $\mathbb{C}$ . Then conditions (E1)–(E4) are satisfied if, and only if,  $\mathbb{C}$  is regular. When  $\mathbb{C}$  is regular, then condition (E5) holds if, and only if,  $\mathbb{C}$  is Mal'tsev (see [3, 5]). Condition (M) is certainly satisfied if  $\mathbb{C}$  is Barr-exact, but there are many examples of regular categories  $\mathbb{C}$  in which (M) is valid but which fail to be Barr-exact (for instance, the quasi-variety of torsion-free abelian groups, or the category of topological groups, both of which are also Mal'tsev).

As mentioned in the introduction, a full reflective subcategory  $\mathbb{X}$  of  $\mathbb{C}$  is a *Birkhoff* subcategory if it is closed in  $\mathbb{C}$  under subobjects and regular quotient objects which, according to [23], is equivalent to the condition that for every regular epimorphism  $f: A \rightarrow B$ , the square (F) is a pushout square of regular epimorphisms. Notice that any pullback square of regular epimorphisms in a regular category is necessarily a pushout, and that this implies that also every member of the class  $\mathcal{E}^1$  must be a pushout square. When  $\mathbb{C}$  is Barr-exact and Mal'tsev, one has the converse: every pushout square of regular epimorphisms is in  $\mathcal{E}^1$  (by Theorem 5.7 of [8]) hence every Birkhoff subcategory is strongly  $\mathcal{E}$ -Birkhoff. In particular when  $\mathbb{C}$  is a Mal'tsev variety then condition (B) is satisfied if, and only if,  $\mathbb{X}$  is a subvariety.

Hence the list of conditions (E3)–(E5), (M) and (B) is satisfied in particular in the situation studied in [13], i.e. for  $\mathbb{C}$  a semi-abelian (hence exact Mal'tev) category,  $\mathbb{X}$  a Birkhoff subcategory of  $\mathbb{C}$  and  $\mathcal{E}$  the class of all regular epimorphisms in  $\mathbb{C}$ . This includes the prototypical example where  $\mathbb{C}$  is the variety of groups and  $\mathbb{X}$  is the variety of abelian groups. Some other examples are the following::

**Example 3.2.** As just explained, the conditions (E3)–(E5), (M) and (B) are satisfied for  $\mathbb{C}$  any exact Mal'tsev category,  $\mathbb{X}$  any Birkhoff subcategory of  $\mathbb{C}$  and  $\mathcal{E}$  the class of all regular epimorphisms in  $\mathbb{C}$ . For instance, we may choose  $\mathbb{X}$  to be the Birkhoff subcategory of *abelian objects*: objects  $A \in \mathbb{C}$  which admit a (necessarily unique) internal Mal'tsev operation (=a morphism  $p: A \times A \times A \rightarrow A$  satisfying  $p(x, x, y) = y$  and  $p(x, y, y) = x$ ).

An example of this situation outside the scope of [13] is the following: for a fixed set  $B$ , let  $\mathbb{C}$  be the category of (small) groupoids with set of objects  $B$ , whose morphisms are the functors  $F: G \rightarrow H$  such that  $F(b) = b$  for every object  $b \in B$ . Then  $\mathbb{C}$  is indeed exact Mal'tsev (it is even *protomodular*—see [2]), but not pointed, hence not semi-abelian, unless  $B$  is a one-element set, in which case  $\mathbb{C}$  is isomorphic to the variety of groups. The subcategory  $\mathbb{X}$  of abelian objects of  $\mathbb{C}$  consists exactly of those groupoids whose groups of automorphisms are abelian (see Theorem 4 in [4]).

**Example 3.3.** Another example of a non-pointed exact Mal'tsev category  $\mathbb{C}$  with a Birkhoff subcategory  $\mathbb{X}$  is the variety  $\mathbb{C}$  of commutative rings (with unit) with its subvariety  $\mathbb{X}$  of Boolean rings (=rings  $A$  such that  $a^2 = a$  for every  $a \in A$ ).

We want to prove that the above conditions imply the assumptions of Theorem 2.8 and are inherited by the induced Galois structure  $\Gamma_1$ . We already know the following:

**Lemma 3.4.** [11, Proposition 3.4] *If the class  $\mathcal{E}$  of morphisms in the category  $\mathbb{C}$  satisfies conditions (E1)–(E5), then the class  $\mathcal{E}^1$  of morphisms of  $\mathbf{Ext}(\mathbb{C})$  satisfies these same conditions.*

Before continuing, we recall that, for any monadic extension  $p: E \rightarrow B$ , the corresponding morphism  $p: (E, p) \rightarrow (B, 1_B)$  in  $(\mathbb{C} \downarrow B)$  is a regular epimorphism. In fact, it follows immediately from this that  $p$  is not only a regular epimorphism in  $(\mathbb{C} \downarrow B)$ , but in *any*  $(\mathbb{C} \downarrow C)$  as soon as a morphism  $g: B \rightarrow C$  exists in  $\mathcal{E}$ . More precisely, we have for any  $g: B \rightarrow C$  in  $\mathcal{E}$  that  $p: (E, gp) \rightarrow (B, g)$  is a regular epimorphism in  $(\mathbb{C} \downarrow C)$ , simply because it is the image of  $p: (E, p) \rightarrow (B, 1_B)$  by the left adjoint functor  $\Sigma_g: (\mathbb{C} \downarrow B) \rightarrow (\mathbb{C} \downarrow C)$ .

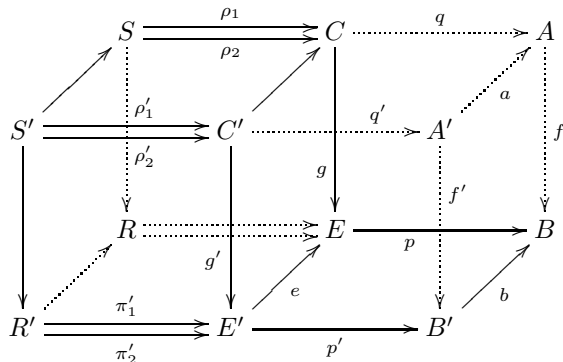
**Lemma 3.5.** *If the class  $\mathcal{E}$  of morphisms in the category  $\mathbb{C}$  satisfies conditions (E1)–(E5) as well as (M), then the class  $\mathcal{E}^1$  of morphisms of  $\mathbf{Ext}(\mathbb{C})$  too satisfies condition (M).*

*Proof.* By Lemma 2.9, the result will follow if we can prove, for any  $(p', p): e \rightarrow b$  in  $\mathcal{E}^1$ , that the functor

$$K^{(p', p)}: (\mathbf{Ext}(\mathbb{C}) \downarrow b) \rightarrow \mathbf{Ext}(\mathbb{C})^{\downarrow \mathbf{Eq}(\mathcal{A})(p', p)}$$

is a category equivalence. We shall do this by constructing an essential inverse for it.

Let us write  $\pi_i: R \rightarrow E$  and  $\pi'_i: R' \rightarrow E'$  ( $i \in \{1, 2\}$ ) for the kernel pair projections of  $p$  and  $p'$ , respectively. Consider an arbitrary object of  $\mathbf{Ext}(\mathbb{C})^{\downarrow \mathbf{Eq}(\mathcal{A})(p', p)}$ ,



Moreover, any morphism in  $\text{Ext}(\mathbb{C})^{\downarrow \text{Eq}(\mathcal{A})(p', p)}$  induces, via the essential inverses for  $K^p: (\mathbb{C} \downarrow B) \rightarrow \mathbb{C}^{\downarrow \text{Eq}(\mathcal{A})(p)}$  and  $K^{p'}: (\mathbb{C} \downarrow B') \rightarrow \mathbb{C}^{\downarrow \text{Eq}(\mathcal{A})(p')}$ , morphisms in  $(\mathbb{C} \downarrow B)$  and  $(\mathbb{C} \downarrow B')$ , and it is easily verified that these determine a morphism in  $(\text{Ext}(\mathbb{C}) \downarrow b)$  (by keeping in mind that  $q': (C', bf'q') \rightarrow (A', bf')$  is a regular epimorphism in  $(\mathbb{C} \downarrow B)$ , for arbitrary objects of  $\text{Ext}(\mathbb{C})^{\downarrow \text{Eq}(\mathcal{A})(p', p)}$  as in the above diagram).  $\square$

We still have to prove that if  $\Gamma$  satisfies conditions (E4), (E5), (M) and (B), then  $\text{Cov}_\Gamma(\mathbb{C})$  is a strongly  $\mathcal{E}^1$ -Birkhoff subcategory of  $\text{Ext}(\mathbb{C})$ . In order to obtain a left adjoint for  $H_1: \text{Cov}_\Gamma(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{C})$ , we verify that the conditions of Theorem 2.8 are satisfied:

**Proposition 3.6.** *If  $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \mathcal{E})$  satisfies conditions (E4), (E5), (M) and (B), then it is admissible.*

*Proof.* As explained in the proof of Proposition 2.6 in [13], this is a consequence of the following two facts: for every morphism  $f: A \rightarrow B$  in  $\mathcal{E}$ , the unit  $\eta_f^B: (A, f) \rightarrow H^B I^B(A, f)$  is a regular epimorphism in the category  $(\mathbb{C} \downarrow B)$  by conditions (B) and (M), and for every object  $B \in \mathbb{C}$ , the functor  $H^B$  reflects isomorphisms because  $H: \mathbb{X} \rightarrow \mathbb{C}$  is fully faithful and  $\eta_*^B: (\mathbb{C} \downarrow HI(B)) \rightarrow (\mathbb{C} \downarrow B)$  is monadic, once again by conditions (B) and (M). Indeed, for an arbitrary  $(X, \varphi) \in (\mathbb{X} \downarrow I(B))$ , the

triangular identity

$$H^B(\epsilon_\varphi^B) \circ \eta_{H^B(X, \varphi)}^B = 1_{H^B(X, \varphi)}$$

tells us that the (regular epic) adjunction unit  $\eta_{H^B(X, \varphi)}^B$  is a split monomorphism, hence an isomorphism. Consequently,  $H^B(\epsilon_\varphi^B)$ , and therefore also  $\epsilon_\varphi^B$  is an isomorphism, which proves that  $H^B$  is fully faithful.  $\square$

Note that any commutative square **(E)** of morphisms in  $\mathcal{E}$  may be considered as a commutative square in  $(\mathbb{C} \downarrow B)$  in an obvious way. If the square is a pullback (in  $(\mathbb{C} \downarrow B)$  or, equivalently, in  $\mathbb{C}$ ) and all four of its morphisms are monadic extensions (hence, regular epimorphisms in  $(\mathbb{C} \downarrow B)$ ) then it is necessarily also a pushout (in  $(\mathbb{C} \downarrow B)$ ). Consequently, as soon as a square **(E)** of monadic extensions is in  $\mathcal{E}^1$ , it is a pushout square in  $(\mathbb{C} \downarrow B)$ . This will be useful to prove the following proposition:

**Proposition 3.7.** *If  $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \mathcal{E})$  satisfies conditions (E4), (E5), (M) and (B), then  $I: \mathbb{C} \rightarrow \mathbb{X}$  preserves those pullbacks **(A)** for which  $f, g \in \mathcal{E}$  and  $g$  is a split epimorphism.*

*Proof.* Let us write  $\text{Ext}^2(\mathbb{C})$  for the full subcategory of the “double arrow” category  $\text{Arr}^2(\mathbb{C}) = \text{Arr}(\text{Arr}(\mathbb{C}))$  determined by the class  $\mathcal{E}^1$ , and  $\mathcal{E}^2$  for the class  $(\mathcal{E}^1)^1$ . By Lemma 3.4, first applied to the pair  $(\mathbb{C}, \mathcal{E})$  and next to the pair  $(\text{Ext}^2(\mathbb{C}), \mathcal{E}^1)$ , we have that every split epimorphism in  $\text{Ext}^2(\mathbb{C})$  lies in the class  $\mathcal{E}^2$ .

Now, consider the diagram

$$\begin{array}{ccccc}
 & & HI(D) & \xrightleftharpoons{\quad} & HI(A) \\
 & \nearrow \eta_D & \downarrow & & \nearrow \eta_A \\
 D & \xrightarrow{\quad} & A & & \\
 \downarrow & & \downarrow f & & \downarrow \\
 & \nearrow \eta_C & HI(C) & \xrightleftharpoons{\quad} & HI(B) \\
 C & \xrightarrow{\quad g \quad} & B & & \\
 & \searrow \eta_B & & & 
 \end{array}$$

in which the front square is a pullback for which  $f, g \in \mathcal{E}$  and  $g$  is a split epimorphism, and the back square is its image by the functor  $HI: \mathbb{C} \rightarrow \mathbb{C}$ . By the strong  $\mathcal{E}$ -Birkhoff property of  $\mathbb{X}$ , the left and right hand squares are in  $\mathcal{E}^1$ , hence we may consider the diagram as a split epimorphism in the category  $\text{Ext}^2(\mathbb{C})$ , whence a morphism in the class  $\mathcal{E}^2$ . It follows that the induced commutative square

$$\begin{array}{ccc}
 D & \xrightarrow{\quad} & HI(D) \\
 \cong \downarrow & & \downarrow \\
 C \times_B A & \longrightarrow & HI(C) \times_{HI(B)} HI(A)
 \end{array}$$

is in  $\mathcal{E}^1$ . In particular, it is a pushout in  $(\mathbb{C} \downarrow HI(C) \times_{HI(B)} HI(A))$ . Hence, the right hand vertical morphism is an isomorphism because the left hand one is so by assumption.

This proves that the back square of the cube is a pullback. Since  $H$  is a full inclusion, the result follows.  $\square$

Hence, we have that the conditions of Theorem 2.8 hold as soon as  $\Gamma$  satisfies conditions (E4), (E5), (M) and (B), in which case  $H_1: \text{Cov}_\Gamma(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{C})$  admits a left adjoint  $I_1: \text{Ext}(\mathbb{C}) \rightarrow \text{Cov}_\Gamma(\mathbb{C})$ . We still have to prove that the square **(F)** is in  $\mathcal{E}^1$ , for every  $f: A \rightarrow B$  in  $\mathcal{E}$ . For this, we first show that every reflection unit is in  $\mathcal{E}^1$ . In fact, for this weaker assumptions on  $\Gamma$  suffice:

**Lemma 3.8.** *If  $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \mathcal{E})$  satisfies the assumptions of Theorem 2.8 as well as conditions (E4) and (E5), and if every reflection unit  $\eta_A: A \rightarrow HI(A)$  is in  $\mathcal{E}$ , then every reflection unit  $\eta_f^1: f \rightarrow I_1(f)$  is in  $\mathcal{E}^1$ .*

*Proof.* Consider a morphism  $f: A \rightarrow B$  in  $\mathcal{E}$ , and Diagram **(C)**, where  $\bar{f} = I_1(f)$ . We must prove that the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\eta_f^1} & \bar{A} \\ f \downarrow & & \downarrow \bar{f} \\ B & \xlongequal{\quad} & B \end{array}$$

is in  $\mathcal{E}^1$ , for which it clearly suffices to show that  $\eta_f^1: A \rightarrow \bar{A}$  is in  $\mathcal{E}$ . Note that the square

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\pi_1} & A \\ \eta_{A \times_B A} \downarrow & & \downarrow \eta_A \\ HI(A \times_B A) & \xrightarrow{HI(\pi_1)} & HI(A) \end{array}$$

is in  $\mathcal{E}^1$  by condition (E5) and the assumption that the reflection units are in  $\mathcal{E}$ . This means, in particular, that in the subdiagram

$$\begin{array}{ccccc} A \times_B A & \xrightarrow{q} & P_0 & \xrightarrow{\bar{\pi}_1} & A \\ \pi_2 \downarrow & & p \downarrow & & \downarrow f \\ A & \xrightarrow{\eta_f^1} & \bar{A} & \xrightarrow{\bar{f}} & B \end{array}$$

of Diagram **(C)** we have that  $q \in \mathcal{E}$ . Since the right hand square is a pullback,  $f \in \mathcal{E}$  implies  $p \in \mathcal{E}$ , and we can conclude from Conditions (E3) and (E4) that also  $\eta_f^1: A \rightarrow \bar{A}$  lies in  $\mathcal{E}$ , as desired.  $\square$

**Lemma 3.9.** *If  $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \mathcal{E})$  satisfies conditions (E4), (E5), (M) and (B), then  $\Gamma_1 = (\text{Ext}(\mathbb{C}), \text{Cov}_\Gamma(\mathbb{C}), I_1, H_1, \eta^1, \mathcal{E}^1)$  satisfies (B):  $\text{Cov}_\Gamma(\mathbb{C})$  is a strongly  $\mathcal{E}^1$ -Birkhoff subcategory of  $\text{Ext}(\mathbb{C})$ .*

*Proof.* Let us write  $\text{Ext}(\mathbb{X})$  for the full subcategory of the arrow category  $\text{Arr}(\mathbb{X})$  determined by the morphisms  $\varphi: X \rightarrow Y$  in  $\mathbb{X}$  for which  $H(\varphi) \in \mathcal{E}$ , and  $\bar{H}$  for the obvious extension of  $H: \mathbb{X} \rightarrow \mathbb{C}$  to  $\text{Ext}(\mathbb{X}) \rightarrow \text{Ext}(\mathbb{C})$ . Since  $HI(\mathcal{E}) \subseteq \mathcal{E}$ , we also have that  $I: \mathbb{C} \rightarrow \mathbb{X}$  extends to a functor  $\bar{I}: \text{Ext}(\mathbb{C}) \rightarrow \text{Ext}(\mathbb{X})$ . Clearly,  $\bar{H}$  is fully faithful just as  $H$ , and  $\bar{I}$  is left adjoint to  $\bar{H}$ . We moreover have that  $\bar{H}(\mathcal{E}^1) \subseteq \mathcal{E}^1$ :

indeed, consider for any morphism  $a \longrightarrow b$  in the class  $\mathcal{E}^1$  the commutative square

$$\begin{array}{ccc} a & \xrightarrow{\bar{\eta}_a} & \bar{H}\bar{I}(a) \\ \downarrow & & \downarrow \\ b & \xrightarrow{\bar{\eta}_b} & \bar{H}\bar{I}(b) \end{array}$$

induced by the reflection unit; since from Lemma 3.4 we know that the class  $\mathcal{E}^1$  satisfies conditions (E3) and (E4), we need only note that  $\bar{\eta}_b \in \mathcal{E}^1$  by the strong  $\mathcal{E}$ -Birkhoff property of  $\mathbb{X}$ .

Hence, we have a Galois structure  $\bar{\Gamma} = (\text{Ext}(\mathbb{C}), \text{Ext}(\mathbb{X}), \bar{H}, \bar{I}, \bar{\eta}, \mathcal{E}^1)$  which, moreover, satisfies the assumptions of Lemma 3.8: indeed, from Lemmas 3.4 and 3.5 we know that conditions (E4), (E5) and (M) are satisfied, and since pullbacks in  $\text{Ext}(\mathbb{C})$  along morphisms in  $\mathcal{E}^1$  are pointwise pullbacks in  $\mathbb{C}$ ,  $\bar{\Gamma}$  is easily verified to satisfy also the remaining assumptions. Furthermore, we clearly have that a morphism  $(f', f): a \longrightarrow b$  in the class  $\mathcal{E}^1$  is a normal extension (=covering) with respect to  $\bar{\Gamma}$  if, and only if, both  $f$  and  $f'$  are normal extensions (=coverings) with respect to  $\Gamma$ , and the left adjoint  $\bar{I}_1: \text{Ext}(\text{Ext}(\mathbb{C})) \longrightarrow \text{Cov}_{\bar{\Gamma}}(\text{Ext}(\mathbb{C}))$  to the inclusion functor  $\bar{H}_1: \text{Cov}_{\bar{\Gamma}}(\text{Ext}(\mathbb{C})) \longrightarrow \text{Ext}(\text{Ext}(\mathbb{C}))$  is obtained simply by pointwise application of  $I_1: \text{Ext}(\mathbb{C}) \longrightarrow \text{Cov}_{\Gamma}(\mathbb{C})$ .

Applying Lemma 3.8 to  $\bar{\Gamma}$  now yields the result.  $\square$

Finally, by combining Lemma's 3.4, 3.5 and 3.9, Propositions 3.6 and 3.7, and Theorem 2.8, we obtain our main result:

**Theorem 3.10.** *For every Galois structure  $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \mathcal{E})$  satisfying conditions (E4), (E5), (M) and (B) (and (E1)–(E3) and (G)), there exists a Galois structure  $\Gamma_1 = (\mathbb{C}_1, \mathbb{X}_1, I_1, H_1, \eta^1, \mathcal{E}^1)$ , once again satisfying these conditions, such that  $\mathbb{C}_1$  is the full subcategory of  $\text{Arr}(\mathbb{C})$  corresponding to  $\mathcal{E}$ ,  $\mathbb{X}_1$  is its full subcategory consisting exactly of the coverings with respect to  $\Gamma$ , and  $\mathcal{E}^1$  is as in Proposition 3.1.*

Hence, by induction, there exists for every  $n \geq 1$  a Galois structure  $\Gamma_n = (\mathbb{C}_n, \mathbb{X}_n, H_n, I_n, \eta^n, \mathcal{E}^n)$  such that  $\mathbb{C}_n$  is the full subcategory of the “ $n$ -fold arrow” category  $\text{Arr}^n(\mathbb{C}) = \text{Arr}(\text{Arr}^{n-1}(\mathbb{C}))$  corresponding to  $\mathcal{E}^{n-1}$ ,  $\mathbb{X}_n$  is its full subcategory consisting exactly of the coverings with respect to  $\Gamma_{n-1}$ , and  $\mathcal{E}^n = (\mathcal{E}^{n-1})^1$ . (Here we assumed that  $\mathbb{C}_0 = \mathbb{C}$ ,  $\mathbb{X}_0 = \mathbb{X}$ ,  $\mathcal{E}^0 = \mathcal{E}$  and  $\text{Arr}^0(\mathbb{C}) = \mathbb{C}$ .)

Moreover, we have for each  $\Gamma_n$  that every morphism in  $\mathcal{E}^n$  is a monadic extension and that every covering is a normal extension.

**Remark 3.11.** Note that a chain of Galois structures  $\Gamma_n = (\mathbb{C}_n, \mathbb{X}_n, H_n, I_n, \eta^n, \mathcal{E}^n)$  may exist, with  $\mathbb{C}_n = \text{Ext}(\mathbb{C}_{n-1})$ ,  $\mathcal{E}^n = (\mathcal{E}^{n-1})^1$  and where  $\mathbb{X}_n$  consists of the coverings with respect to  $\Gamma_{n-1}$ , even if the assumptions of Theorem 3.10 are not satisfied. To see this, take  $\mathbb{C}$  arbitrary, put  $\mathbb{X} = \mathbb{C}$  and choose  $\mathcal{E}$  to be the class of all morphisms in  $\mathbb{C}$ . Then condition (M) is not satisfied, but the existence of the Galois structures  $\Gamma_n$  is readily observed. Note that a covering with respect to  $\Gamma_n$  is simply an  $n + 1$ -fold morphism.

Or, we may choose  $\mathcal{E}$  to be the class of regular epimorphisms in, say, an exact category  $\mathbb{C}$ , in which case condition (M) is satisfied, and still take  $\mathbb{X} = \mathbb{C}$ . In this case, we do not only have the existence of the Galois structures  $\Gamma_n$ , but the last part of the theorem is satisfied as well: every morphism in  $\mathcal{E}^n$  is a monadic extension and



the classes of coverings and of normal extensions with respect to  $\Gamma_n$  coincide (and, in fact, coincide with  $\mathcal{E}^n$ , as well as with the class of trivial coverings). However, condition (E5) fails for any  $\mathbb{C}$  which is not Mal'tsev.

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